A PRACTICAL PRESENTATION OF SOME PROBLEMS CORRELATING MATHEMATICS AND PHYSICS.[1]

EVERY student of social conditions and movements is wont to fix upon some phase of human activity which for each epoch may serve to epitomize the dominating idea of the epoch. But any idea which possesses sufficient human significance to occupy the center of the stage for an era must manifest its influence on human affairs in various ways. Each student must construe these manifestations under his own phase-angle. Out of this necessity arise both the weakness and the power of the individual. But if each would do his full duty to his fellows, he must communicate to them the benefit of his point of view. In this consists his power. In his failure to do this lies his weakness.

To the educationist the distinguishing mark of our age is, perhaps, the general and intense interest taken in popular education. A salient characteristic of educated American citizenship is its profound faith—amounting almost to credulity—in the notion that public education will accomplish almost any and every human good. The student of education would probably tell us that the dominant notes of current educational thought are mutual helpfulness and mutual serviceableness; in short, that the watchword of education is no longer "competition," but "co-operation."

Co-operation—individual, civic, and national—has already wrought among us wonders great enough to summon educational students to our shores to ascertain what it is in our school system that contributes to our national prosperity and prestige.

It is our faith in the efficacy of co-operation as a means of educational advance that calls us together in this conference. It is this same desire to co-operate that calls for a statement in words of some of the results of our attempts to correlate mathematics with physical science.

[1] Read before the Mathematics Section of the Central Association of Science and Mathematics Teachers, 1903.

To state one's impelling motives and ideals in mathematical pedagogy often calls for a terminology which is in danger of misleading a general gathering of teachers. It is difficult to speak of the laboratory plan of mathematics without conveying the impression that the speaker is wedded to some form of mechanical device for teaching mathematics. It is, however, only another instance of the difficulty of speaking briefly and concretely without becoming too concrete. The writer therefore wishes to disclaim, in the use of the phrase "laboratory plan of mathematics," the possession of any desire save that of securing brevity by the use of a phrase which has of late acquired at least a degree of explicitness among us.

Perhaps no word in our language has suffered so serious a perversion of meaning among mathematicians as has the word "method". During the last twenty-five years pedagogical sentiment has swung from knowledge of subject-matter to the acquisition of certain professional devices as the sole qualification for teaching. But these extreme views have in divers ways been reduced to a *reductio ad absurdum*.

There is no place where a study of method is more needed or would be more propitious of improvement than in the high school. Many mathematics teachers, clearly convinced of the dangers of allowing artificial devices for securing pedagogic ends to serve as makeshifts for a sound knowledge of subject-matter, and equally convinced of the need of a thoughtful adaptation of subject-matter to modern conditions and to immature pupils, find themselves halting between two extreme views, uncertain in which direction to throw their influence, lest they be trapped into the absurdities of the extremist, no matter which route they take. That this state of mind is explicable in the light of recent conditions, anyone conversant with these conditions will admit, though he would scarcely condone it. Still, however much we dislike the word "method," we do not fail in our respect for methodical work. But that the fear of being dragged too far even in the right direction should paralyze us into inaction is certainly unfortunate, if not puerile. For it is with mathematical movements as with moral—we cannot stand

still. If we do not advance, relatively to the movements of the age at least, we must recede. A prominent educator recently reminded us that most of us reach the high-water mark of professional service about three years after beginning practical teaching. Furthermore, mathematical movements, like vibrations of the pendulum, advance by vibrating back and forth through the position of stable equilibrium.

But in the midst of these oscillatory movements, which may easily be tolerated, if they meet the one fundamental condition of eventuating in progress, let us at all times keep in view the supreme end to be attained. It is trite to say that as mathematical teachers we are interested in the laboratory method, the correlation method, the individual method, the natural method, only as means to the accomplishment of something greater and of more fundamental significance than any, or all, of these mere methods taken together. Whatever interest attaches to these methods is, of course, altogether a secondary interest, borrowed from the interest in the primary central aim. While perhaps each earnest student of the problem of better teaching would state differently his conception of this central and dominating aim, it is nevertheless well that each attempt a statement of it, since each statement must be tinged with the individual point of view. By a sort of summation, or rather, of integration, of these individual conceptions we may in time hope to arrive at the correct general notion.

No conscientious believer in advance can allow himself to be debarred from giving his point of view by the dread of being trapped into foolish extremes through a misconstruction of his ideas. It is out of some such considerations as these that the writer is led to attempt a statement of what the problem really is which is consciously or unconsciously subsumed in all our efforts looking to the unification of mathematical subjects in the high school. To put the meaning simply, all the mathematical methodology of which we have of late been guilty—and, thanks to the live mathematical teachers, it has been considerable—has had the fundamental purpose of making the high-school boy, or girl, feel the true worth of mathematical study.

It is a great mistake to imagine that boys and girls turn away from mathematical studies because these studies are too difficult. When made to feel the real worth of performing it, the most difficult task becomes to them easy and palatable. Difficulty and danger do not debar youthful aspirants to a place on the high-school team. Stated differently, it is the seeming unprofitableness of the mathematical tasks imposed by current mathematics teaching that makes these subjects repellant to young men and women. As a teacher and lover of mathematics, I do not see why the boys and girls should be chided for neglecting the artificial tasks so rife among us, set by current high-school mathematical texts, and tolerated and dignified by too many secondary teachers. Young persons show the best of judgment in repudiating much of the work given them. To eschew the foreign substance demonstrates not only that they still possess sufficient mental springiness to reject matter having little or no affinity with their thought-processes, but also that relevant and assimilable subject-matter may still be taken up by their intellectual systems and used. As the writer sees the situation, the laboratory plan, the individual method, the correlation plan, are all of them merely practical means of getting the solution of the tremendously difficult problem, forced upon us by circumstances which the schoolroom formalists—the routinists—have at least aided in bringing into existence, viz., the problem of deartificializing the matter and method of mathematics. The frayed-out abstractions of mathematical scholasticism—for that is what the excessively abstract work seems to young students—are being pretty effectually shed by high-school boys and girls who are vitally in touch with the life that now is; and though some teachers are pretty thoroughly convinced of the inadequacy of such subject-matter, they are not yet quite certain what sort of subject-matter will secure the much-desired ends of mathematical education. Hence it comes about naturally that some who have set about in search of a better approach to abstract mathematics than the customary formal one find that high-school science is rich in useful mathematical matter, inasmuch as high-school students quite generally regard science work as being important enough to repay their

efforts to learn it. But what do we behold? Someone suggests the science work as a good basis for starting the mathematics. Then the inevitable critic says that the suggester is advocating the laboratory plan; and so in the mind of the critic a method bug-a-boo is born, christened, and thrust upon us, quite innocently, even unconsciously, on our part. Someone is then in grave danger of being blamed with having started a new method, and somehow the critic is nowhere in evidence when the fixing of guilt is being attempted. But, in the light of past pedagogic experience, we ought to be able to maintain our equanimity, even if new methods do arise in mathematics occasionally. Aye, I suggest that we ought even to try them out in the class-room before dissipating too much bad sentiment and good energy in quarreling with the so-called "new departure." New methods, new suggestions, are always most disturbing to the formalist, but let us remember that advance often occurs through them. Of course, most complaint is always heard where most breaking of images occurs; but this is usually where there exists the greatest need of disturbance.

This much by way of introduction may be pardoned on the score of a distinct desire on my part not to be thought of either as an innovator or as an apostle, or even as an advocate of any new method as such. The laboratory plan is not the concern of this paper. Still I would hasten to confess to the possession of a very zealous desire to see mathematics teaching everywhere, but particularly in the secondary school and in the early collegiate years, vitalized and brought into inspiring touch with the interests, motives, and convictions of young men and women. I confess to a feeling of dissatisfaction at seeing high-school and early college students turning away from every abstract mathematical subject not absolutely required and rushing, pell-mell, to almost anything as a welcome relief. I am very far from believing this tendency to be merely the manifestation of a general desire on the part of all college students to seek the so-called "snap" courses. I have never felt the solace that some college teachers seem to find in this view. The very fact that these young persons are in high school and college makes

the assumption safe that some, even many, of them are persons of serious purpose. Many are working their way through college, and such do not thus play at "fast and loose" with what they believe to be their best interests. They say they omit the optional mathematics courses because they see no earthly use for these subjects.

Again, the general ignorance of the average citizen of liberal education of mathematical truth is little short of appalling. Ask such a person, after he has unblushingly confessed himself to be a "mathematical ignoramus," how it happened that he didn't give more attention to mathematics while in college, and the answer is usually: "I didn't think the subject important then." Someone, the ignoramus himself chiefly perhaps, was to blame for this, but incidentally, I believe, the mathematical representatives he came in contact with are also, in a measure, culpable. Most of us would have difficulty in recalling many of our former mathematical instructors who took much pains to point out to us the great importance in almost any pursuit of a good knowledge of mathematics and to back it up with a little first-hand evidence. Most of us had to content ourselves with a "You'll see some day." As the writer passes in review the rather long line of glittering icebergs that have had to do with his academic career in mathematics, he can find two or three, out of some thirty, who could and did say something besides "mental discipline" to the favor of mathematical study. Doubtless others were more fortunate than the reader, but I dare say the vast majority that are in mathematical work for the love of it discovered their liking for it and their respect for it more or less accidentally and quite independently of their high-school, or college, instructors of mathematics. Let us not argue that, since we have become so worthy under this *laissez faire* and accidental treatment, it is therefore necessarily best for all students.

My point of view is that instructors of mathematics, not in any didactic way, but through subject-matter making a strong appeal to the understanding and judgment of young men and women, undertake to show the purpose, relation, and value of mathematical study to the all-around concerns and interests of

modern life. My thought is that this can be done most effectually in the high school by basing the mathematical work upon the pupil's former experience, upon his science, and upon whatever other quantitative questions appeal to him as of sufficient intrinsic importance to present a real problem to his mind.

Finally, I believe that physics makes a strong and natural demand for mathematics, but also that it is but *one* phase of the whole mathematical demand of the high school. It is for other persons and occasions to present the full-orbed mathematical demand of the high school. My task today is the humbler one of giving you a presentation of only some problems correlating mathematics and physics.

But, having confessed several things touching my views of the mathematical situation in the secondary school, I wish also to add a few items of my creed as pertaining to this same situation. I hold these truths to be self-evident:

1. That the results of current mathematics teaching are altogether satisfactory to no one.

2. That the reasons for this dissatisfaction are numerous and capable of varied diagnosis.

3. That mathematics teachers are partly to blame for poor results.

4. That mathematics teachers are not altogether to blame.

5. That science teachers, school superintendents, and machine-like administrative officers are aiders and abetters—even fosterers—of current shortcomings of mathematics teaching.

6. That the "cock-sure" state of mind and the general self-satisfaction with one's own output toward which all mathematics teachers gravitate must bear a good deal of the odium of popular censure.

7. That it requires more sympathy, patience, industry, humility, knowledge, and tact to be a successful secondary mathematics teacher than to be a successful minister of the gospel.

8. That divine providence is also involved in the controversy for not making mathematics pupils always to remember accurately everything they were ever taught, and for making mathematics teachers so imperfect.

9. That mathematics teaching might even be made to furnish results satisfactory to physics teachers and still be very far from what it should be.

10. That the aim of mathematics teaching should be, not to exemplify or illustrate any method of teaching, but to underlay the necessary amount of abstract work with such a basis of useful material for thought that mathematical truth and method may appeal to the student as a very powerful aid to the right understanding of this real world.

11. That blind and disjointed mathematical study—that is, work lacking point and purpose to the student—at the very best can have but very low educational value.

12. That, of all secondary subjects, mathematics most indispensably demands unity and clear purpose.

13. That the most serious criticism against current mathematics teaching is that it appeals to the pupil as mere empty abstraction and barren ingenuity.

I now pass to a few practical examples, such as I have actually used with high-school classes with what seemed to me a sufficient measure of success to justify doing much more of it myself and commending to those concerned in the practical problem of unifying the mathematics and science in the secondary curriculum.

I. A method of handling experimental data in a mathematical class may be exemplified here by the study of friction. The data and results of such a study are given in the table for the friction of pine on pine:

Experiment with Carriage on Edge	Combined Load, L	Observed Force, F_o	Compared Friction, F_c	Error, $o-c_1$	Compared E	Error, $o-c_2$
1	500 g.	160 g.	162.5	− 2.5	141.5	+18.5
2	700	210	227.5	−17.5	214.9	− 4.9
3	900	280	292.5	−12.5	288.3	− 8.3
4	1,100	370	357.5	+13.5	361.7	+ 8.3
5	1,300	420	422.5	− 2.5	435.1	+15.1
6	1,500	500	487.5	+12.5	509.0	− 9.0
7	1,700	600	552.5	+47.5	581.9	+18.1
8	1,900	650	617.5	+32.5	655.3	− 5.3
9	2,100	730	682.5	+47.5	728.7	+ 1.3
10	2,300	800	747.5	+52.5	802.1	− 2.1
11	2,500	875	812.5	+62.5	875.5	+ 0.5

Casting the eye down columns 2 and 3, it is readily seen that F increases when L increases. The first suggestion would then be that L and F increase proportionally. Show the pupil that this is equivalent to saying that the ratio $F:L$ is the same throughout; *i. e.*, that $\frac{F}{L}=k$, a constant.

Now find k thus: In the second experiment divide F $(=210)$ by L $(=700)$, and get $k_2=0.30$. Also find k_{10} similarly from the tenth experiment $800\div 2{,}300=0.35$. Suppose the true value, k, to be $(k_2+k_{10})\div 2=0.325$, or that the law of friction is

$$F\times 0.325L. \qquad (1)$$

Now, the values of L given by experiment were as they are given in column 2. Substitute these values successively in law (1) and obtain the numbers, F_c, of column 4. If law (1) had been correct, and both observations and computations had been made correctly and accurately, the corresponding F_o's and the F_c's would have been equal. We form the errors $(o—c)$, in the sense $F_o—F_c$, noting carefully the signs, and enter them in column 5. We assume also that our law should be such as to make the errors as small as possible.

Casting the eye down the column of $(o-c)$'s, we readily see that the errors increase as L increases. Both the algebraic signs and the numeral values show this. What we should like to do with law (1), then, is so to alter it as to make F larger when L is large, and, at the same time, not to change the values of F too much for small values of L. This will be done by taking the law in the form—

$$F=xL+y.$$

Now let us find x and y. To secure a little greater accuracy, we take for L and F the mean of their values in experiments 2 and 3, and substitute in (2), getting $254=800x+y$. Similarly, 9 and 10 give $765=2{,}200x+y$. Solving these equations for x and y, we find $x=0.367$ and $y=-42$, giving the law—

$$F_o=0.367\,L-42. \qquad (2)$$

Substituting in law (2) for L the numbers of column 2, we find successively those of column 6 for F'. The errors, $o-c_2$,

are again formed, entered in column 7, and compared with those of column 5. The representation is found to be very much better and for values of L lying between 500 g. and 2,500 g. may be regarded as satisfactory. The corrected law (2) is then

$$F = 0.367\ L - 42\,. \qquad (2)$$

There is little of the algebraic technique, from the beginning of algebra to the solution of simultaneous linear equations, that is not called for in this work, though, at every step, it appeals to the student as subsidiary—*incidental*, if you please, but not *accidental*—to an appreciably valuable main purpose. Nor is the arithmetic ignored. The student is also learning where "to put the point," for he is handling decimals throughout.

II. The pine block, when placed on its side, and similarly loaded and dragged along the pine surface, gave the values of the following table, and (1) the approximate, (2 and 3), two corrected, and (4) the mean law are stated in order just below the table:

Exp.	L	F_o	F_c	$F_o - F_c$	F_2	$F_o - F_2$	F_3	$F_o - F_3$	F_4	$F_o - F_4$
1	500 g.	130 g.	150 g.	− 20 g.	105 g.	+ 25 g.	127 g.	+ 3 g.	116 g.	+ 14 g.
2	900	250	270	− 20	249	+ 1	255	− 5	252	− 2
3	1,300	375	390	− 15	393	− 18	383	− 8	388	− 13
4	1,800	540	540	0	573	− 33	543	− 3	558	− 18
5	2,100	675	630	+ 45	681	− 6	639	+ 36	660	+ 15
6	3,500	825	750	+ 75	825	0	767	+ 58	850	+ 5

$F = 0.30\ L$ (approximate law from experiments 1 and 6). (1)
$F = 0.36\ L - 75$ (first corrected law from experiments 2 and 6). (2)
$F = 0.32\ L - 33$ (second corrected law from experiments 1 and 4). (3)
$F = 0.34\ L - 54$ (third corrected law, mean of (2) and (3)). (4)

Law (1) gave the values of column F_c; law (2) those of column F_2; law (3) those of column F_3; and law (4) those of column F_4. A study of the error columns, showing the evidence as to the character of the successive representations, should be made. It is also worth while to give the student the method of judging as to the nature of the representation by a comparison of the sum of the squares of the errors in each case.

Graphical work might profitably come in here in either of two ways: (1) the student might represent geometrically the final law; (2) he might have plotted the observations, and, with the aid of a taut string, might have drawn the mean line and derived

the law independently from the picture, and then have compared the law, graphically derived, with that derived algebraically and arithmetically as above. But I would not advise using the graph of observations, or of equations, otherwise than as a means to a better understanding of some law, or truth, which is really illumined by it. I doubt the wisdom of using the graph for its own sake much before the early college years. I am convinced of the wisdom of using it *as a means* to "lighting up" difficult matters even before the high school.

The law of the pendulum may be used in the algebra class to introduce the study of proportion and variation—two subjects which should be taught together.

III. (1) A pendulum, 75 cm long, was made to vibrate, first on a short arc, then on a long arc, the number of vibrations in 30 seconds in the two cases being 35 and 35. This convinced the class there was no need of paying attention to the length of the arc of vibration.

(2) Then an experiment showed that the weight of the bob made no special difference in the time of the vibration.

(3) Six pendulums, of lengths given in the second column of the table below, were allowed to vibrate for 30 seconds, each member of the class counting the vibrations. The average of the counts gave the several numbers of column 3. The time of vibrations were then computed as in column 4, as also the square roots of the lengths as given in column 5.

Different members of the class then computed the ratios of assigned pairs of the times and of the corresponding lengths, and compared the two ratios. The two ratios were found so nearly equal that it was readily seen that, with absolutely correct counts and measures, the ratios would be exactly equal. The law of proportionality $t_1 : t_2 = \sqrt{l_1} : \sqrt{l_2}$ was then formulated and studied algebraically.

Exp.	l	n	t	$\sqrt{l}$	k
1...........	19.1 cm	68.5	0.43	4.37	0.1
2...........	24.8	60.0	0.50	4.99	0.1
3...........	37.1	49.4	0.61	6.93	0.1
4...........	74.6	35.0	0.86	8.94	0.1
5...........	103.8	29.4	1.02	10.19	0.1
6...........	224.0	20.8	1.44	14.96	0.1

The class was then asked to find to two decimals and shorten to one the ratio of each t to its $\sqrt{l}$. In each case this was found to be 0.1, and it was then seen that the following law could be written:

$$\frac{t}{\sqrt{l}} = k \text{ or } t = k\sqrt{l}$$

where k is a constant, nearly equal to 0.1. The class was then told that the correct $k = \frac{\pi}{\sqrt{g}}$, where $g = 9.81$ meters, and k was found from this relation also, for the practice it gave in reducing such equations.

The law $t = \pi\sqrt{\frac{l}{g}}$ was then formulated, and the class was asked to compute the length of the seconds pendulum. If time had permitted, the value of g for various places would have been given, and the lengths of the seconds pendulum as well as the time of vibration of the meter, half-meter, one-yard, one-foot pendulums for each case would have been computed for the sake of the drill.

IV. A light steel spring was loaded with weights, W, as shown in column 1 and the lengths, L, of the spring under the several loads were as given in column 2. The stretches, S, corresponding to the loads, are given in column 3.

W	L	S	k
0 g.	72.3 mm	....	....
25.	76.1	3.8 mm	6.58
50.	79.8	7.5	6.67
75.	83.6	11.3	6.64
100.	87.4	15.1	6.62

Shortened to one decimal place, k is 6.6 and may be regarded as constant for the spring, so that the law may be formulated thus: $W = kS$, where $k = 6.6$, for this spring. The law may, and should, also be derived in the form

$$\frac{W_1}{W_2} = \frac{S_1}{S_2} \;.$$

V. The lengths, l, and heights, h, of an inclined plane on which

a car carrying loads, L, was supported by the forces, F, are tabulated here:

l	h	L	F
100 cm	6.18 cm	638 g	38 g
100	6.18	1,038	63
100	12.5	638	78
100	12.5	1,038	123
100	25.5	638	153
100	25.5	1,038	255

The ratios $h : l$ and $F : L$ were computed, compared, and the law formulated into both

$$F : L = h : l$$

and
$$F = kL, \ k \text{ being } \frac{h}{l} .$$

Here the k is constant for any loading, but variable for different loadings, and this extension of the idea *fixed constant* to the notion of *arbitrary constant* is important and is greatly facilitated by this experiment.

An experiment on Boyle's Law with a flexible mercurial tube gave the following data for teaching the law $p_1 : p_2 = v_2 : v_1$ (*inverse variation*), and also suggested the form $pv =$ a constant, for algebraic study:

p	v	$p \times v$
46.0 cm	20.8	1,566
83.8	18.9	1,584
98.5	16.1	1,586

Several other laws were given and treated experimentally, but the foregoing will suffice to give an idea of the character of the work.

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